PHYSICAL REVIEW B **81**, 041311(R) (2010)

Interaction-induced edge channel equilibration

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(Received 14 October 2009; revised manuscript received 4 January 2010; published 28 January 2010)

The electronic distribution functions of two Coulomb-coupled chiral edge states forming a quasi-one-dimensional system with broken translation invariance are found using the equation of motion approach. We find that relaxation and thereby energy exchange between the two edge states is determined by the shot noise of the edge states generated at a quantum point contact (QPC). In close vicinity to the QPC, we derive analytic expressions for the distribution functions. We further give an iterative procedure with which we can compute numerically the distribution functions arbitrarily far away from the QPC. Our results are compared with recent experiments.

DOI: 10.1103/PhysRevB.81.041311 PACS number(s): 73.23.−b, 73.43.Cd, 72.70.+m

Two decades ago, edge states[1] (ESs) were demonstrated to be a physical reality by creating a nonequilibrium population[2] through selective injection and detection of carriers in different states along the same edge.[3–5] Experiments revealed that the inter-edge carrier scattering could be strongly suppressed[3–5] over distances of 80 μm. Now in a series of experiments the group of Pierre[6,7] has investigated the nonequilibrium distribution function in an ES as it evolves along a channel away from a QPC at which it is initially created. Following metallic diffusive conductors[8] and carbon nanotubes[9] this is only the third type of mesoscopic conductor for which the distribution in the presence of transport has been measured. It is of the highest interest since it is the first distribution measurement on a single quantum channel.

The experiments are carried out in a high mobility two-dimensional electron gas at a filling factor $\nu=2$ such that there is an outer (spin-up) nonequilibrium ES and an inner (spin-down) equilibrium ES. The distribution function is measured with the help of a quantum dot (QD) sufficiently small to provide transmission only through a single resonant level (see Fig. 1). The QD serves as an energy spectrometer and permits the reconstruction of the distribution function in the outer ES.

The experiments reveal two surprising features: First, the initial nonequilibrium distribution created at the QPC and calculated from noninteracting scattering theory differs only weakly from the measured one over distances of close to 1 μm.[6,7] At large distances from the QPC, due to the Coulomb interaction between carriers in the two ESs, the distribution function evolves into an equilibrium distribution function at an effective electrochemical potential and temperature. The outer nonequilibrium ES transfers part of its energy to the inner ES. The two ESs equilibrate toward the same equilibrium distribution with the same temperature (but still at different electrochemical potentials due to lack of particle exchange between the two ESs). The second surprise of the experiments is the fact that the temperature of the distribution functions at large distance in the two ESs is *lower* than dictated by equilibrium thermodynamic arguments.[7] The first surprise shows that relaxation due to inter-ES interaction is weak. The second surprise implies that equilibration occurs not only between the inner and outer ESs but that there must be an additional equilibration mechanism which cools the two ESs below what would be expected from inter-ES coupling alone. We propose that additional excitations in the bulk,[10] which couple predominantly to the inner ES, have to be considered to understand this effect. Although the nature of these excitations remains unclear, the experimental findings of Ref. 7 are consistent with this hypothesis. For example, it is found that when the inner ES is forced to form a short closed loop, then relaxation in the outer ES is strongly suppressed.

The physics of ESs is often discussed within the framework of bosonization theory, where the elementary excitations have bosonic character and are of collective nature.[11] Also the experiment by le Sueur *et al.*[7] can be discussed in this manner.[12] In contrast, we take the weak equilibration seen at distances of less than a micrometer as the starting point of a discussion which treats inter-ES interaction perturbatively.[13] The interaction is described in terms of two-body collisions. We use the equation of motion approach for second quantized operators to derive an evolution equation for the distribution functions which resembles a Boltzmann collision term with the added complication that there are two different initial distributions (one for each ES). Alternatively the Coulomb matrix elements which appear in this theory

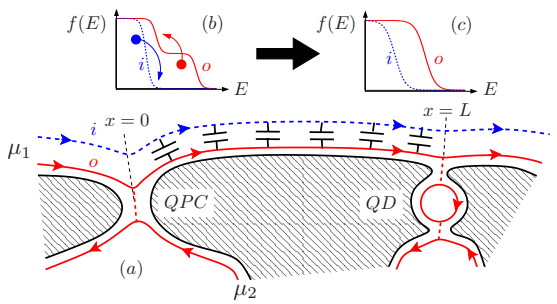

FIG. 1. (Color online) (a) The experimental setup to measure the electronic distribution function of an ES. The full (red) curve represents the measured outer ES while the dashed (blue) curve represents a copropagating inner ES. The two ESs exchange energy via Coulomb interaction between $x=0$ and $x=L$. The initial distribution functions (b) relax via energy-conserving particle-hole excitation processes toward Fermi functions (c).

can be taken from a random-phase approximation (RPA) theory[14] in which the electron densities in each channel fluctuate and interact through an (effective) capacitance. To treat equilibration at longer distances, we iterate numerically the solution for short distances. At large distances the distribution functions approach their equilibrium form dictated by entropy maximization.

We describe the ESs in terms of scattering states $\chi_{\alpha E}(x)$ with energy E and label $\alpha=o,i$ (i: inner and o: outer). The inter-ES interaction is given by

$$H_{\rm int}=\frac{1}{2}\sum_{\alpha}\int dEdE' U_{\alpha}(E',E)a^{\dagger}_{\alpha E'}a_{\alpha E}, \tag{1}$$

where $a^{\dagger}_{\alpha E}(a_{\alpha E})$ is the creation (annihilation) operator for the scattering state $\chi_{\alpha E}$ and $U_{\alpha}(E',E)$ is the potential operator for scattering a particle from E to E' in the ES α at the expense of a particle scattering in the opposite ES $\bar{\alpha}$. Explicitly $U_{\alpha}(E_{1'},E_1,t)=\int dE_2dE_{2'}V^{\alpha\bar{\alpha}}_{E_{1'}E_{2'},E_1E_2}a^{\dagger}_{\bar{\alpha}E_{2'}}(t)a_{\bar{\alpha}E_2}(t)$ in the Heisenberg picture and $V^{\alpha\bar{\alpha}}_{E_{1'}E_{2'},E_1E_2}$ is the inter-ES electron-electron interaction matrix element for the scattering process $(\alpha E_1,\bar{\alpha}E_2)\rightarrow(\alpha E_{1'},\bar{\alpha}E_{2'})$. Using the Heisenberg equation of motion $i\hbar\partial_t a_{\alpha E}(t)=[a_{\alpha E}(t),H]$, the electronic distribution function $f_{\alpha}(E)$ in ES α can be found by evaluating $\langle a^{\dagger}_{\alpha E}(t)a_{\alpha E'}(t)\rangle=\delta(E-E')f_{\alpha}(E)$. The noninteracting distributions are $f^0_i(E)=f^0_{\mu_i}(E)$ and $f^0_o(E)=\mathcal{R}f^0_{\mu_1}(E)+\mathcal{T}f^0_{\mu_2}(E)$, where $f^0_{\mu}\equiv\{1+\exp[(E-\mu)/k_BT]\}^{-1}$ and $\mathcal{T}$ ($\mathcal{R}$) is the transmission (reflection) probability of the QPC (see Fig. 1). The chemical potential of the inner ES μ_i can experimentally be tuned independently of μ_1 and μ_2 by using an additional QPC (not shown in Fig. 1). To second order in the interaction matrix element the distribution is $f^{(2)}_{\alpha}=f^0_{\alpha}+\delta f^{(2)}_{\alpha}$, where (see supplementary material[15] for details)

$$\begin{aligned}\delta f^{(2)}_{\alpha}(E)&=2\pi\int_{-\infty}^{\infty}d\omega\Big[f^0_{\alpha}(E+\hbar\omega)[1-f^0_{\alpha}(E)]S^a_{\delta U_{\alpha}\delta U_{\alpha}}(E,E+\hbar\omega,\omega)\\&\quad-f^0_{\alpha}(E)[1-f^0_{\alpha}(E+\hbar\omega)]S^e_{\delta U_{\alpha}\delta U_{\alpha}}(E+\hbar\omega,E,\omega)\Big].\end{aligned} \tag{2}$$

The first term contains the absorption potential fluctuation spectrum[16] $S^a_{\delta U_{\alpha}\delta U_{\alpha}}(E,E',\omega)$ describing an absorption of energy $\hbar\omega$ by the ES $\bar{\alpha}$, while the ES α goes from energy E' to E. Likewise the second term with the emission fluctuation spectrum $S^e_{\delta U_{\alpha}\delta U_{\alpha}}$ describes the emission of energy $\hbar\omega$ from the ES $\bar{\alpha}$ to the ES α, which consequently leads to the transition $E\rightarrow E+\hbar\omega$ in α. The fluctuation spectra are to lowest order in the interaction and defined by $2\pi\delta(\omega+\omega')\times S^a_{\delta U_{\alpha}\delta U_{\alpha}}(E',E,\omega)\equiv\langle\delta U_{\alpha}(E,E',\omega)^{(1)}\delta U_{\alpha}(E',E,\omega')^{(1)}\rangle$, where $\delta U^{(1)}_{\alpha}\equiv U^{(1)}_{\alpha}-\langle U^{(1)}_{\alpha}\rangle$ is the Fourier transformed operator for the deviation from the average potential to first order in the interaction. The emission spectrum is found by interchanging the two δU_{α} in the absorption spectrum or equivalently by changing the sign of ω. Explicitly, the spectra are found to be

$$S^a_{\delta U_{\alpha}\delta U_{\alpha}}(E',E,\omega)=h\int dE''|V^{\alpha\bar{\alpha}}_{E'E''+\hbar\omega,EE''}|^2f^0_{\bar{\alpha}}(E'')[1-f^0_{\bar{\alpha}}(E''+\hbar\omega)], \tag{3a}$$

$$S^e_{\delta U_{\alpha}\delta U_{\alpha}}(E',E,\omega)=h\int dE''|V^{\alpha\bar{\alpha}}_{E'E'',EE''+\hbar\omega}|^2f^0_{\bar{\alpha}}(E''+\hbar\omega)[1-f^0_{\bar{\alpha}}(E'')], \tag{3b}$$

where the interpretation in terms of emission and absorption spectra is clear. By inserting these into Eq. (2) the similarity with the collision integral in the Boltzmann equation becomes evident.

Next we wish to calculate $\delta f^{(2)}_{\alpha}(E)$. To this end, the inter-ES scattering process $(\alpha E_1,\bar{\alpha}E_2)\rightarrow(\alpha E_{1'},\bar{\alpha}E_{2'})$ needs to be considered. If the ESs are perfectly translation invariant, then energy *and* momentum conservation together reduce the available one-dimensional phase space enormously compared to higher dimensions.[17] This leads us to consider the more realistic non-translation-invariant case caused by the fact that the ESs follow the equipotential lines created by the sample edges and the impurity potential. Including this nontranslation invariant ES physics leads to the presence of nonmomentum conserving scattering processes increasing the phase space substantially.[18,19] The broken translation invariance is included into the model of the inter-ES interaction matrix element $|V^{\alpha\bar{\alpha}}_{E_{1'}E_{2'},E_1E_2}|^2$. To avoid modeling a specific geometry we perform a statistical average over the geometry of the ESs and thereby introduce the momentum breaking correlation length ℓ_p, which is smaller than the size of the region of relaxation L. For simplicity, an effective interaction of the form $V(x,x')=\delta(x-x')g(x)$ is used and it is assumed that the deviation of $g(x)$ from some mean value g_0 is Gaussian distributed, i.e., $\overline{(g(x)-g_0)(g(x')-g_0)}=A/(\sqrt{2\pi}\ell_p)\exp[-(x-x')^2/(2\ell_p^2)]$, where $A/(\sqrt{2\pi}\ell_p)$ is the maximal deviation and $\overline{\cdots}$ denotes the geometrical averaging. This yields an interaction with a momentum conserving and a momentum breaking part. The latter is (see supplementary material[15] for details)

$$\overline{|V^{\alpha\bar{\alpha}}_{E_{1'}E_{2'},E_1E_2}|^2}_{\Delta k\neq 0}=\frac{AL}{h^4v_{\alpha}^2v_{\bar{\alpha}}^2}\exp[-(\Delta k\ell_p)^2/2], \tag{4}$$

where $\Delta k=(E_1-E_{1'})/(\hbar v_{\alpha})+(E_2-E_{2'})/(\hbar v_{\bar{\alpha}})$, using linear dispersion relations with different velocities v_{α} for the two ESs. Note that for linear dispersions with *different* velocities there is no phase space for scattering in the momentum conserving limit, $\Delta k=0$, but in the very special (almost pathological) case $v_{\alpha}=v_{\bar{\alpha}}$, momentum and energy conservation are equivalent leading to plenty of phase space. The specific model for the interaction and the matrix element is not of great importance as long as it includes the physics leading to nonmomentum conserving processes, which in turn introduces a new length scale ℓ_p.

For energy conserving scattering, the model matrix element Eq. (4) only depends on the transferred energy in the scattering[20] since $\Delta k=\omega(1/v_{\alpha}-1/v_{\bar{\alpha}})$. This means that the energy integral in the fluctuation spectra of Eq. (3) can be

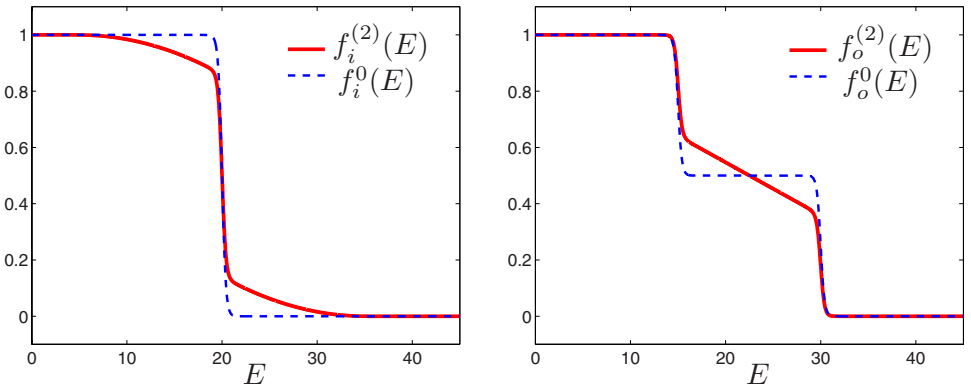

FIG. 2. (Color online) Analytically calculated inner (left) and outer (right) ES distribution functions in the regime $k_BT, |\mu_2-\mu_1| \ll |\Delta E|$. The parameters are (energies in μeV): $\mathcal{T}=0.5$, $\mu_1=15$, $\mu_2=30$, $\mu_i=20$, and $k_BT=0.2$.

done analytically upon which it becomes evident that $\delta f_\alpha^{(2)}(E) \propto \mathcal{T}(1-\mathcal{T})$. Thus the greater the shot noise of the QPC, the faster the relaxation is. The elementary scattering processes leading to relaxation consist of a particle loosing energy in the noisy outer ES and a particle gaining energy in the noiseless inner ES as illustrated on Fig. 1(b). The matrix element introduces a new energy scale $\Delta E \equiv (\hbar/\ell_p) v_\alpha v_{\bar{\alpha}}/(v_{\bar{\alpha}}-v_\alpha)$, which limits the possible amount of energy transferred between the two ESs in the scattering process since the matrix element is proportional to $e^{-(\hbar\omega/\Delta E)^2}$. In the limit that $k_BT, |\mu_2-\mu_1| \ll |\Delta E|$ the distribution functions for the inner and outer ES can be found analytically to be

$$\delta f_o^{(2)}(E) = -\gamma^2 \mathcal{T}(1-\mathcal{T})(\mu_2-\mu_1) \times [f^0_{\mu_2}(E) - f^0_{\mu_1}(E)]\left[E - \frac{1}{2}(\mu_1+\mu_2)\right], \quad (5)$$

$$\delta f_i^{(2)}(E) = \frac{\gamma^2}{2}\mathcal{T}(1-\mathcal{T}) \times \left\{-[f^0_{\mu_i}(E) - f^0_{\mu_i^-}(E)][(\pi k_BT)^2 + (E-\mu_i^-)^2] + [f^0_{\mu_i^+}(E) - f^0_{\mu_i}(E)][(\pi k_BT)^2 + (E-\mu_i^+)^2]\right\}, \quad (6)$$

where $\gamma^2 \equiv (2\pi)^2 AL/[h^4 v_\alpha^2 v_{\bar{\alpha}}^2]$ and $\mu_i^\pm = \mu_i \pm (\mu_2-\mu_1)$ is the maximal and minimal energy of particles affected by the scattering process in the inner ES. Here it is seen that the maximal available energy (apart from thermal excitations of order k_BT) is given by the energy difference $\mu_2-\mu_1$ creating the step distribution. The scattering processes create a linear slope on the plateau of the distribution of the noisy outer ES as shown in Fig. 2. The slope crosses the middle of the plateau and it is proportional to the noise of the QPC and the energy available $\mu_2-\mu_1$. The inner noiseless distribution gets a tail on both sides of the Fermi level, which extends over the length of the plateau $\mu_2-\mu_1$. In the general case, the distribution functions can be found numerically and the matrix elements in Eq. (4) have to be included in the calculation, but the transferred energy is still limited by ΔE.

The above perturbative results apply for a short distance L after the QPC and express the distribution functions at L in terms of the (unperturbed) distribution functions at the origin. Once the distribution functions at L are known we can use them to calculate the distribution functions at a distance $2L$ via Eq. (2). By iterating this procedure we can thus describe the effective length dependence of the energy relaxation. A convenient quantity with which to characterize the relaxation of f_α at temperature T is given by the *excess temperature* $T_{exc,\alpha}$ (Ref. 6) defined as

$$k_BT_{exc,\alpha} \equiv \sqrt{\frac{6}{\pi^2}\int dE \Delta f_\alpha(E)(E-\tilde{\mu}_\alpha) - (k_BT)^2}. \quad (7)$$

Here $\Delta f_\alpha(E) = f_\alpha(E) - \theta(\tilde{\mu}_\alpha - E)$ is the difference between the actual distribution function and a zero-temperature Fermi distribution with the same number of particles and hence $\tilde{\mu}_\alpha = E_0 + \int_{E_0}^\infty dE f_\alpha(E)$, where E_0 is chosen such that $f_\alpha(E)=1$ for $E<E_0$. $k_BT_{exc,\alpha}$ gives the energy of the nonthermal excitations in f_α. The initial excess temperature right after the QPC of the inner ES is zero and the one of the outer ES is given by $k_BT^0_{exc,o} = \{(3/\pi^2)\mathcal{T}(1-\mathcal{T})\}^{1/2}|\mu_2-\mu_1|$. Because of energy conservation, $\Sigma_\alpha T_{exc,\alpha}$ is a conserved quantity in the equilibration process. Furthermore due to entropy maximization the excess energy is distributed equally among the two ESs, which in the limit of long distances thus converge toward Fermi distributions with equal excess temperatures given by $k_BT^{\infty(2)}_{exc} = \{(3/2\pi^2)\mathcal{T}(1-\mathcal{T})\}^{1/2}|\mu_2-\mu_1|$. The excess temperature of the outer ES measured in Ref. 7 does indeed saturate at large distances toward a finite value. This value is however found to be systematically lower than the above prediction for large voltage biases.[15] Surprisingly it agrees well with the value $k_BT^{\infty(3)}_{exc} = \{\mathcal{T}(1-\mathcal{T})\}^{1/2}|\mu_2-\mu_1|/\pi$ expected from energy equipartition among *three* instead of only two channels. What could provide the additional relaxation channel? Excitation of internal modes of the inner ES has been suggested as an additional relaxation mechanism.[12] In Ref. 7 it has been observed that if the inner ES is forced to form a short enough closed loop, such that the energy level spacing of its (discrete) spectrum is larger than the available energy provided by the voltage bias $\mu_2-\mu_1$, then relaxation of the outer ES is strongly suppressed. Thus internal modes of the outer ES are not excited. Motivated by this observation, we suggest instead, that there exist excitations of localized states in the bulk,[10] which are coupled via Coulomb interaction to both the inner ES and an ES on the opposite side of the sample. As long as the bulk excitations can be created, such a mechanism would allow extra energy to be carried away from the outer ES.

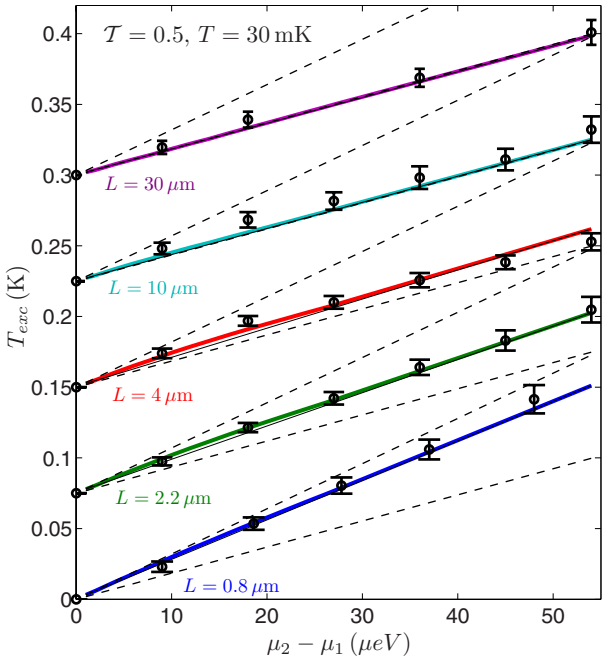

FIG. 3. (Color online) The excess temperature of the outer ES versus the voltage difference across the QPC. For clarity curves for different lengths have been shifted upward by 75 mK. The dashed (black) lines indicate the initial value $T^0_{exc,o}$ (upper lines) and the asymptotic value $T^{\infty(3)}_{exc}$ (lower lines) as expected from energy equipartition. The thin (black) lines are guides for the eyes to better see the weak nonlinearity present at intermediate distances (especially for 2.2 and 4 μm). Experimental data (circles with error bars) courtesy of Pierre *et al.*

As a first approach we model this extra degree of freedom as an additional ES coupled to the inner ES only, initially in equilibrium at the electronic temperature, which we take to be $T=30$ mK. This then contributes an extra collision term in Eq. (2) and allows a quantitative comparison with the experiment.[7] The fitting procedure is detailed in Ref. 15 and the result is shown in Fig. 3. The best fit is obtained when the coupling strength to the bulk excitations is about three times larger than the inter-ES coupling strength and when $\Delta E = 14.3$ μeV, which for v_o and v_i between 10^4 and 10^5 m/s leads to $\ell_p \geq 0.5$ μm. For intermediate distances (i.e., 2.2 and 4 μm) both the data and our numerics display a similar weakly nonlinear behavior of the excess temperature as a function of the voltage bias.

To conclude, we have computed the distribution functions of two Coulomb-coupled ESs in the integer quantum Hall regime. We derived an analytic expression for the leading-order correction to the noninteracting theory, which is present in a system without translation invariance. We have shown further that the result obtained in the long-distance limit, by iterating the perturbative solution numerically, is in quantitative agreement with a recently performed experiment if we take into account the electric coupling between the innermost ES and bulk excitations. Finally we note that measurements of the distribution functions of both channels for even shorter distances than 0.8 μm should allow for further critical testing of our theory.

The authors would like to thank H. le Sueur, C. Altimiras, and F. Pierre for fruitful discussion and for sharing their data. This work was supported by the Swiss NSF.

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